Two correlation patterns as indicators for underlying dynamics of complex systems

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(Received 26 January 2004; published 12 January 2005)

Two spatial correlation patterns, fixed-to-arbitrary and neighboring bin correlation patterns, are suggested. It is demonstrated that these patterns present scale-independent and distinguishable measures for correlated and uncorrelated random multiplicative cascade processes. Their application to very high multiplicity events in relativistic heavy ion collisions is discussed.

DOI: 10.1103/PhysRevE.71.017103 PACS number(s): 89.75.Fb, 05.45.−a, 13.85.Hd, 25.75.−q

Correlations have been proven to be a powerful tool in exploring the underlying dynamics of many complex systems. They relate directly to the interaction mechanism. For example, the momentum correlation between identical particles gives a direct access to the spatial distribution of the emitting source [1], the power law behavior of the correlation strength with diminishing correlation length implies a self-similar multifractal [2,3], and so on.

The simplest correlation case is two-bin correlation, defined by [4]

$$C_{K_1,K_2} = \langle p_{K_1} p_{K_2} \rangle - \langle p_{K_1} \rangle \langle p_{K_2} \rangle, \tag{1}$$

where K_1 and K_2 are the positions of two bins in the phase space and $p(K)$ is the probability in the Kth bin. The average is over all events in the sample. In the conventional investigation of correlations, the measure is usually limited to the relation between correlation strength and correlation length with an additional average over all pairs of bins of the same distance in the whole observed domain [2–4]. This average procedure (usually referred to as a horizontal average [5]) obviously smooths out the space structure of the correlations. This structure is sensitive to different underlying dynamics and allows us to investigate a complex system on a deeper level.

A straightforward space-position related correlation is the three-dimensional correlation pattern, i.e., C_{K_1,K_2} vs K_1, K_2. However, the structure of this kind of three-dimensional pattern is very complicated and the underlying dynamics is hard to observe thereby; cf., Fig. 1 of Ref. [6]. A two-dimensional or a projection of the three-dimensional pattern will be able to show the fine structure much more clearly. There are numbers of way to construct a two-dimensional correlation pattern. A good one should include as much information on space correlations as possible and be simple as well.

In this Brief Report, we suggest two two-dimensional correlation patterns, which contain important information on the underlying dynamics. As an example, we demonstrate how they can be used as indicators for correlated and uncorrelated random multiplicative cascade processes. The possibility of their application in very high multiplicity events of multiparticle production is discussed at the end of the paper.

An important and interesting piece of information is the change of correlation strength with space distance. It can be obtained by fixing one bin K_1 and varying the other one K_2. This is the *fixed-to-arbitrary* bin correlation pattern. It contains correlation information at various lengths. Another important piece of information is how the neighboring bins are correlated to each other, i.e., the *neighboring* bin correlation pattern. In the following, we will show how these two patterns provide us distinguishable information about correlated and uncorrelated random multiplicative cascade processes.

A random multiplicative cascade process (RMCP) is the simplest example of having a well-defined multifractal structure and has been used extensively in various fields. For example, the fragmentation processes in turbulence [2,7] and successive branching in the QCD parton shower [8] are both this kind of process.

The main idea of such a process is simply a series of self-similar random cascades in spatial partitions. It is generally described as follows. At the first step of the cascade, a given initial interval with length Δ and unit probability $p_0 = 1$ is split into τ subdivisions ("sub-bins"). The probability for each sub-bin in splitting is determined by a weight w which is distributed according to a *splitting function* $p(w_1, w_2, \ldots, w_\tau)$. Then each bin obtained from the first generation of the cascade is again independently split into τ bins and so on. The probability distribution for such a process with $\tau = 2$ is depicted in Table I.

After $\nu = 1, 2, \ldots, J$ generations, the final number of bins is τ^ν, and the probability in a specific bin j_ν is a product of ν w's over all previous generations: $p^\nu_{j_\nu} = w_{1j_1} w_{2j_2} \cdots w_{\nu j_\nu}$, where $j_\nu = 1, 2, \ldots, \tau^\nu$. The distribution of $p^\nu_{j_\nu}$ obtained in this way will fluctuate violently from bin to bin (cf. Fig. 6 in [2]), and has a well-defined multifractal structure.

For an uncorrelated RMCP, the splitting function can be factorized as

$$p(w_1, w_2, \ldots, w_\tau) = p(w_1) p(w_2) \cdots p(w_\tau); \tag{2}$$

therefore, the correlation between any two sub-bins in any elementary splitting is zero. At the first step, $C_{K_1,K_2} = 0$ for

TABLE I. The probability distribution of binary random cascade processes with cascade generation down to $\nu = 2$.

$\nu=0$	$p^0_0 = 1$			
$\nu=1$	$p^1_1 = p^0_0 w_{11}$		$p^1_2 = p^0_0 w_{12}$	
$\nu=2$	$p^2_1 = p^1_1 w_{21}$	$p^2_2 = p^1_1 w_{22}$	$p^2_3 = p^1_2 w_{23}$	$p^2_4 = p^1_2 w_{24}$

any K_1,K_2. At the last step, $C_{K_1,K_2}=0$ if and only if the two bins K_1 and K_2 in Eq. (1) are separated at the very beginning, i.e., at $\nu=1$. In the fixed-to-arbitrary bin correlation pattern, these two bins are $K_1=1$ and $K_2=\tau^{\nu-1}+1,\tau^{\nu-1}+2,\ldots,\tau^{\nu}$; while they are $K_1=\tau^{\nu-1}$ and $K_2=\tau^{\nu-1}+1$ in a neighboring bin correlation pattern. On the contrary, two other bins, having a common ancestor at $\nu>0$, will have a nonvanishing correlation. The strongest correlations are between those two bins split at the last step of the cascade. Hence the smallest correlation in this kind of process is zero.

On the other hand, the splitting function cannot be factorized as Eq. (2) for a correlated RMCP and the correlations in any elementary splitting will not vanish, being above zero for a positively correlated RMCP and below zero for a negatively correlated one.

Therefore, the basic difference between correlated and uncorrelated RMCPs can be clearly observed by the above-suggested two correlation patterns. This conclusion is obviously splitting-number (τ) and scale (Δ/τ^{ν}) independent.

The above arguments can be shown analytically for concrete models if we take the logarithm of the probability instead of the probability itself in Eq. (1) [6]. It can be simply written as

$$C_{K_1,K_2}=\langle \ln p_{K_1}\ln p_{K_2}\rangle-\langle \ln p_{K_1}\rangle\langle \ln p_{K_2}\rangle = A(J-d)+B(1-\delta_{d,0}), \tag{3}$$

where $d=J-\sum_{j=1}^{J}\delta_{K_1^1,K_2^1}\cdots\delta_{K_1^j,K_2^j}$ is the ultrametric distance, which is a measure of how many generations one has to move up before a common ancestor is found for two given bins K_1 and K_2;

$$A=\left.\frac{\partial^2 Q[\lambda_1,\lambda_2]}{\partial\lambda_1^2}\right|_{\lambda=0},\quad B=\left.\frac{\partial^2 Q[\lambda_1,\lambda_2]}{\partial\lambda_1\partial\lambda_2}\right|_{\lambda=0} \tag{4}$$

are respectively the so-called "same-lineage" cumulant, which is the correlation of the common ancestor of K_1 and K_2, and the "splitting" cumulant, which measures the correlation between the two parts split first from their common ancestor; and

$$Q[\lambda_1,\lambda_2]=\ln\left[\int dw_1dw_2p(w_1,w_2)e^{(\lambda_1\ln w_1+\lambda_2\ln w_2)}\right] \tag{5}$$

is the branching generating function (BGF).

There is a class of uncorrelated RMCP models, such as the famous α model [9] with splitting function [6]

$$p_\alpha(w_1,w_2)=\frac{1}{4}[\delta(w_1-(1+\alpha))+\delta(w_1-(1-\alpha))]\times[\delta(w_2-(1+\alpha))+\delta(w_2-(1-\alpha))], \tag{6}$$

where α is a fixed parameter in the region $[-1,1]$. The splitting function of this kind of model can also be parametrized by the binomial, lognormal, or Beta distribution for each weight w [10].

The simplest correlation in random cascade processes comes from the constraint on probability or energy conservation in each splitting step. The well-known p model [11] is one of this kind of process. Its splitting function [6] is

$$p(w_1,w_2)=\frac{1}{2}[\delta(w_1-(1+\beta))+\delta(w_1-(1-\beta))]\times\delta(w_1+w_2-2), \tag{7}$$

where β is a fixed parameter in the region $[-1,1]$ and probability conservation $w_1+w_2=2$ holds in every splitting. Another example of this kind of model is the c model [12], where the w's are random number distributed in [0, 1] with $w_1=(1+\gamma r)/2$ and $w_2=(1-\gamma r)/2$, r is a random number in the interval $[-1,1]$, and γ is the parameter of fluctuation strength, $0\leqslant\gamma\leqslant 1$. Obviously, this model is more flexible and closer to a real system with probability conservation. Its splitting function can be written as

$$p(w_1,w_2)=\frac{1}{\gamma}\left[\theta\left(w_1-\frac{1-\gamma}{2}\right)-\theta\left(w_1-\frac{1+\gamma}{2}\right)\right]\times\delta(w_1+w_2-1). \tag{8}$$

For these three models, by simply inserting their splitting functions into Eq. (5) and then into Eq. (4), their corresponding "same-lineage" and "splitting" cumulants become

$$A_\alpha=\frac{1}{4}\left[\ln\frac{1+\alpha}{1-\alpha}\right]^2,\quad B_\alpha=0, \tag{9}$$

$$A_p=\frac{1}{4}\left[\ln\frac{1+\beta}{1-\beta}\right]^2,\quad B_p=-\frac{1}{4}\left[\ln\frac{1+\beta}{1-\beta}\right]^2, \tag{10}$$

$$A_c=\frac{1}{\gamma}\int_{(1-\gamma)/2}^{(1+\gamma)/2}(\ln w_1)^2dw_1-\frac{1}{\gamma^2}\left[\int_{(1-\gamma)/2}^{(1+\gamma)/2}\ln w_1dw_1\right]^2,$$

$$B_c=\frac{1}{\gamma}\int_{(1-\gamma)/2}^{(1+\gamma)/2}\ln w_1\ln(1-w_1)dw_1-\frac{1}{\gamma^2}\left[\int_{(1-\gamma)/2}^{(1+\gamma)/2}\ln w_1dw_1\right]^2, \tag{11}$$

respectively.

For α, p, and c models, we depict the fixed-to-arbitrary and neighboring bin correlation patterns in Figs. 1(a) and 1(b), respectively, with the parameters $\alpha=0.4$, $\beta=0.4$, $\gamma=0.8$, $J=6$.

The significant differences among these three models are the values of the smallest correlation strength and the one above it. As shown in the figure, the smallest correlation in the α model is zero. In the p model, the correlation spectrum ranges over positive, zero, and negative values, while for the c model, only positive and negative correlations appear without a zero value.

The smallest correlation strength is equal to the value of B at $d=J$ in Eq. (3) and the one above it is $A+B$ at $J-d=1$. The vanishing of B in the α model is caused by independent splitting, while the negative B for both p and c models means a negative correlation between the two parts of the elementary splitting, which is due to the constraint of probability conservation in each splitting. The zero correlation strength

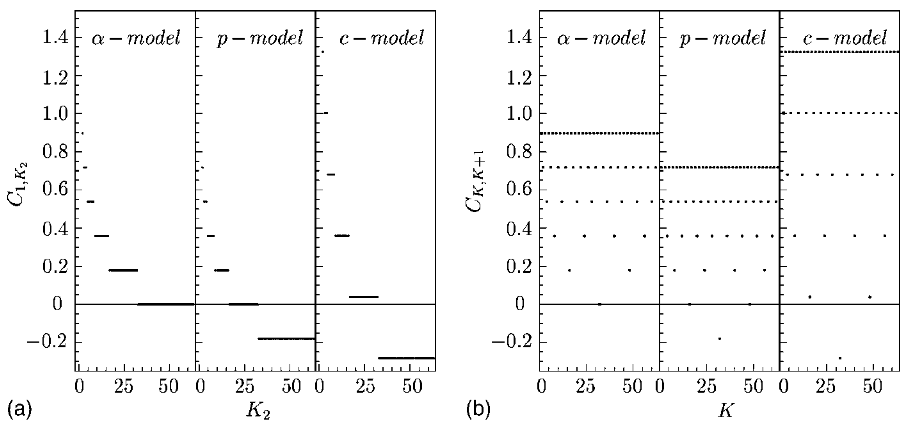

FIG. 1. (a) The fixed-to-arbitrary bin correlation patterns C_{1,K_2} and (b) the neighboring bin correlation patterns $C_{K,K+1}$, of the α, p, and c models. The solids lines in the figures indicate zero correlation strength.

at $J-d=1$ in the p model implies $A_p=-B_p$. This happens to be the case when there is only one mode for possible values of weight factors w in the splitting function, i.e., $w_1=1\pm\beta$ and $w_2=1\mp\beta$. These correspondence relations are valid no matter how many generations the real cascade process is.

So all the observable differences in the patterns come from the different model assumptions. Whether the smallest correlation strength in the suggested two patterns is zero or nonzero provides a scale-independent indication of whether the underlying random multiplicative cascade processes is uncorrelated or correlated. For negatively correlated processes, the correlation strength above the smallest one will be zero (or nonzero) as long as the splitting mode is unique (or nonunique).

In applying the model to multiparticle dynamics, a problem is whether the probability in a small phase space bin can be approximately estimated by

$$p_K \cong n_K/N, \tag{12}$$

where n_K is the number of particles falling into the Kth bin and N is the total number of particles in the event. This equation holds exactly if and only if $N\to\infty$. In current relativistic heavy ion experiments (at the RHIC) [17] the multiplicity of a single event has reached 10^4 [18], and Eq. (13) should be approximately applicable.

In order to show this quantitatively, we choose the above-mentioned c model to do the simulation. The dynamical probabilities $p_1,p_2,\ldots,p_M$ in the $M=2^J=64$ bins are first generated by the model and shown in Fig. 2 as open circles. Then $N=10^4$ particles are put into the $M=64$ bins by Bernoulli distribution. From the simulated n_K, the practically measured p_K is calculated by Eq. (13). Finally, the fixed-to-arbitrary bin correlation pattern is calculated by Eq. (3) and shown in Fig. 2(a) by solid circles. From the figure, we can see that the solid circles already give a regular pattern similar to that of the open ones. The results for $N=10^5$ are depicted in Fig. 2(b), where the solid circles just fall into the open ones, showing that the original pattern is well reproduced. So the suggested measures can be safely applied to high multiplicity events in future heavy ion experiments, such as LHC-ALICE, and are therefore highly recommended.

If a quark-gluon plasma (QGP) is formed in a relativistic heavy ion collision, the correlation between final state particles is expected to extend to a larger space and the correlation strength is greatly suppressed in comparison with those processes without a QGP. In contrast to the correlation pattern of the c model, or existing models for nuclear collisions without the QGP, the resulting correlation pattern will not develop so rapidly with the distance between two bins and will tend to be less dependent on the position of the bins. The suggested correlation patterns just provide a possible identification for those different dynamical evolutions.

In the conventional measures of correlation at a certain correlation length in multiparticle production, a horizontal average over all pairs of bins with the same distance has been taken, e.g., the momentum correlation between identical particles [13] and the balance functions [14], the correlations at different spatial positions are neglected, and so the information contained in neightboring bin correlation pattern, i.e., Fig. 1(b), is lost. The suggested two correlation patterns not only keep this information, but also offer a general measure for it in contrast to the specific position-related correlations used in multiparticle dynamics, such as the forward-backward correlation [15], the correlation between two rapidity windows separated by a specific rapidity gap [16], and so on. Moreover, it is unnecessary to keep the bin size diminishing until it is very small, as is required by

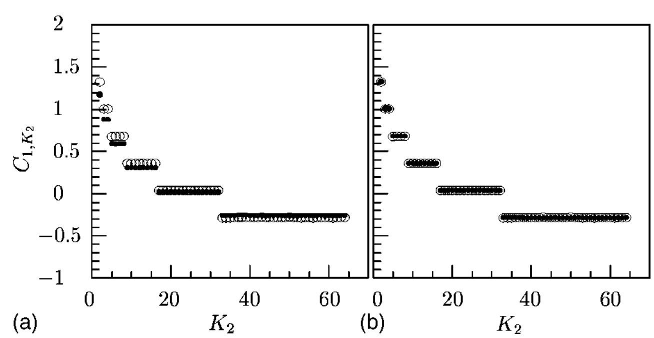

FIG. 2. The fixed-to-arbitrary bin correlation patterns C_{1,K_2}, where the open circles are generated by the c model, the same as those in the third columns of Figs. 1(a) and 1(b), and the solid circles are the results of the c model with finite multiplicity fluctuations (a) $N=10^4$, (b) $N=10^5$.

the anomalous scaling analysis of the correlated factorial moments [3].

In summary, two reduced spatial correlation patterns for two-point correlations are suggested. They are sensitive to the underlying mechanism and work equally well in distinguishing correlated and uncorrelated RMCPs as well as in identifying the unique and nonunique splitting modes in negatively correlated RMCPs. The possibility of their application to multiparticle production is discussed.

We can also find similar reduced spatial correlation patterns in more-than-two-point correlations. Measurements for all of them will be useful in exploring the underlying dynamics of complex system.

We are grateful for the good suggestions of Dr. Nu Xu and Dr. Huangzhong Huang. This work is supported in part by the NSFC under Project No. 90103019 and EDU under Project No. Jiaojisi(2004)295.

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